

Chapter 11 Boundary-value problem for O.D.E.

§ Linear $y'' = p(x)y' + q(x)y + r(x)$, $a \leq x \leq b$, $y(a) = \alpha$, $y(b) = \beta$

1. Finite-difference method

Set $x_0 = a$, $x_{n+1} = b$, $h = (b-a)/(n+1)$, $x_i = x_0 + ih$, $y(x_i) = y_i$,

$$p(x_i) = p_i, \quad q(x_i) = q_i, \quad r(x_i) = r_i$$

$$\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) = p_i \frac{1}{2h}(y_{i+1} - y_{i-1}) + q_i y_i + r_i$$

$$-(1 + \frac{1}{2}hp_i)y_{i-1} + (2 + h^2q_i)y_i - (1 - \frac{1}{2}hp_i)y_{i+1} = -h^2r_i \quad \text{with } y_0 = \alpha, \quad y_{n+1} = \beta$$

$$(2 + h^2q_1)y_1 - (1 - \frac{1}{2}hp_1)y_2 = -h^2r_1 + (1 + \frac{1}{2}hp_1)y_0$$

$$-(1 + \frac{1}{2}hp_i)y_{i-1} + (2 + h^2q_i)y_i - (1 - \frac{1}{2}hp_i)y_{i+1} = -h^2r_i, \quad i = 2 \sim n-1,$$

$$-(1 + \frac{1}{2}hp_n)y_{n-1} + (2 + h^2q_n)y_n = -h^2r_n + (1 - \frac{1}{2}hp_n)y_{n+1},$$

Arranging into the vector form

$$[A]\{Y\} = \{F\}$$

where

$$[A] = \begin{bmatrix} 2 + h^2q_1 & -1 + 0.5hp_1 & 0 & 0 \\ -1 - 0.5hp_2 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & -1 + 0.5hp_{n-1} \\ 0 & 0 & -1 - 0.5hp_n & 2 + h^2q_n \end{bmatrix}, \quad \{Y\} = \begin{Bmatrix} y_1 \\ \bullet \\ \bullet \\ y_n \end{Bmatrix},$$

$$\{F\} = \begin{Bmatrix} -h^2r_1 + (1 + 0.5hp_1)y_0 \\ -h^2r_2 \\ \bullet \\ -h^2r_n + (1 - 0.5hp_n)y_{n+1} \end{Bmatrix}.$$

2. Shooting method

Consider two O.D.Es.

$$y_1'' = p(x)y_1' + q(x)y_1 + r(x), \quad a \leq x \leq b, \quad y_1(a) = \alpha, \quad y_1'(a) = 0$$

$$y_2'' = p(x)y_2' + q(x)y_2, \quad a \leq x \leq b, \quad y_2(a) = 0, \quad y_2'(a) = 1.$$

The solutions $y_1(x)$ and $y_2(x)$ can be obtained from the Runge-Kutta method.

$y = y_1 + cy_2$ satisfies the original differential equation and $y(a) = \alpha$. The

constant c is obtained from $\beta = y(b) = y_1(b) + cy_2(b)$ as

$$c = [\beta - y_1(b)] / y_2(b)$$

If $y_2(b) = 0$, then try $y_2'(a) = 2$ to obtain a new c , and the solution

$y = y_1 + cy_2$ is obtained.

§ Nonlinear $y'' = f(x, y, y')$, $a \leq x \leq b$, $y(a) = \alpha$, $y(b) = \beta$

$$\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) = f[x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1})]$$

$$y_{i+1} - 2y_i + y_{i-1} - h^2 f[x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1})] = 0,$$

$$2y_1 - y_2 + h^2 f[x_1, y_1, \frac{1}{2h}(y_2 - \alpha)] - \alpha = 0,$$

$$-y_{i-1} + 2y_i - y_{i+1} + h^2 f[x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1})] = 0, \quad i = 2 \sim n-1,$$

$$-y_{n-1} + 2y_n + h^2 f[x_n, y_n, \frac{1}{2h}(b - y_{n-1})] - \beta = 0$$

Arranging the equations into a vector form yields

$$\vec{F}(\vec{Y}) = \vec{0}, \quad \text{where } \vec{Y} = \{y_1, y_2, \dots, y_{n-1}, y_n\}^T$$

§ Variation approach

$$\text{Ex. } -\frac{d}{dx}[p(x)\frac{dy}{dx}] + q(x)y = f(x), \quad 0 \leq x \leq l, \quad y(0) = y(l) = 0$$

$$\int_0^l \left\{ -\frac{d}{dx}[p(x)\frac{dy}{dx}] + q(x)y - f(x) \right\} \delta y dx = 0$$

$$\int_0^l \left\{ p(x)\frac{dy}{dx} \frac{d\delta y}{dx} + q(x)y\delta y - f(x)\delta y \right\} dx - p(x)\frac{dy}{dx} \delta y \Big|_{x=0}^{x=l} = 0$$

$$\frac{1}{2} \int_0^l \left\{ p(x)\delta \left(\frac{dy}{dx} \right)^2 + q(x)\delta y^2 - 2f(x)\delta y \right\} dx = 0,$$

$$\delta \int_0^l \left\{ p(x)\left(\frac{dy}{dx} \right)^2 + q(x)y^2 - 2f(x)y \right\} dx = 0.$$

Try a set of functions $\{\phi_1, \phi_2, \dots, \phi_n\}(x)$ with $\phi_i(0) = \phi_i(l) = 0$

$$y(x) = \sum_{i=1}^n c_i \phi_i(x)$$

$$\min_{c_i} I(c_1, \dots, c_n) = \min_{c_i} \int_0^l \left\{ \sum_{i,j=1}^n c_i c_j \left(p \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + q \phi_i \phi_j \right) - \sum_{i=1}^n 2c_i f \phi_i \right\} dx,$$

$$\frac{\partial I}{\partial c_i} = 0 \quad \text{implies} \quad [A]\{c\} = \{b\}$$

$$a_{ij} = \int_0^l \left(p \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + q \phi_i \phi_j \right) dx, \quad b_i = \int_0^l f \phi_i dx$$

$$\text{Ex. } -\frac{d}{dx}[p(x)\frac{dy}{dx}] + q(x)y = f(x), \quad 0 \leq x \leq l, \quad y(0) = a, \quad y(l) = b$$

First, $y_1(x) = a(1 - \frac{x}{l}) + b(\frac{x}{l})$, $y(x) = y_1(x) + y_2(x)$, then $y_2(0) = y_2(l) = 0$

Second, $-\frac{d}{dx}[p(x)\frac{dy}{dx}] + q(x)y = f(x)$ becomes

$$-\frac{d}{dx} \left\{ p(x) \frac{dy_2}{dx} \right\} + q(x)y_2 = f(x) + \frac{b-a}{l} \frac{dp}{dx} - q(x) \left[a(1 - \frac{x}{l}) + b(\frac{x}{l}) \right] \triangleq F(x)$$

Third,

$$\delta \int_0^l \left\{ p(x) \left(\frac{dy_2}{dx} \right)^2 + q(x)y_2^2 - 2F(x)y_2 \right\} dx = 0.$$

Try a set of functions $\{\phi_1, \phi_2, \dots, \phi_n\}(x)$ with $\phi_i(0) = \phi_i(l) = 0$

$$y_2(x) = \sum_{i=1}^n c_i \phi_i(x)$$

$$\min_{c_i} I(c_1, \dots, c_n) = \min_{c_i} \int_0^l \left\{ \sum_{i,j=1}^n c_i c_j \left(p \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + q \phi_i \phi_j \right) - \sum_{i=1}^n 2c_i F \phi_i \right\} dx,$$

$$\frac{\partial I}{\partial c_i} = 0 \quad \text{implies} \quad [A]\{c\} = \{b\}$$

$$a_{ij} = \int_0^l \left(p \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + q \phi_i \phi_j \right) dx, \quad b_i = \int_0^l f \phi_i dx$$

$$\text{Finally, } y(x) = \sum_{i=1}^n c_i \phi_i(x) + y_1(x)$$

$$\text{Ex. } \phi_i(x) = \sin\left(\frac{i\pi x}{l}\right)$$

Finite-Element Method

$$x_0 = 0, \quad x_{2n+1} = l, \quad h = (l-0)/(2n+1), \quad x_i = ih,$$

$$\text{Set } \{x_i, y_i\}_{i=0}^{i=2n+1} \quad \text{with } y_0 = a \quad \text{and } y_{2n+1} = b.$$

$$\text{Generate } y(x) = y_{2i-1} \phi_1^{(i)}(x) + y_{2i} \phi_2^{(i)}(x) + y_{2i+1} \phi_3^{(i)}(x), \quad x_{2i-1} \leq x \leq x_{2i+1}$$

where $\phi_1^{(i)} \sim \phi_3^{(i)}$ are Lagrange's interpolation functions.

The integration $\int_0^l \{p(x)(\frac{dy}{dx})^2 + q(x)y^2 - 2f(x)y\} dx$ can be expressed as

$$\sum_{i=1}^n \int_{x_{2i-1}}^{x_{2i+1}} \{p(x)(\frac{dy}{dx})^2 + q(x)y^2 - 2f(x)y\} dx$$

$$\text{Denote } \bar{y}^{(i)} = \{y_{2i-1} \ y_{2i} \ y_{2i+1}\}^T \quad \text{and} \quad \bar{\Phi}^{(i)}(x) = \{\phi_1^{(i)} \ \phi_2^{(i)} \ \phi_3^{(i)}\}^T(x)$$

Then the integration will be

$$\sum_{i=1}^n \int_{x_{2i-1}}^{x_{2i+1}} \{\bar{y}^{(i)T} (p \bar{\Phi}^{(i)'} \bar{\Phi}^{(i)'}{}^T + q \bar{\Phi}^{(i)} \bar{\Phi}^{(i)T}) \bar{y}^{(i)} - 2 \bar{y}^{(i)T} f(x) \bar{\Phi}^{(i)}\} dx$$

Minimizing the integration yields

$$\sum_{i=1}^n \int_{x_{2i-1}}^{x_{2i+1}} (p \bar{\Phi}^{(i)'} \bar{\Phi}^{(i)'}{}^T + q \bar{\Phi}^{(i)} \bar{\Phi}^{(i)T}) dx \bar{y}^{(i)} - \int_{x_{2i-1}}^{x_{2i+1}} f(x) \bar{\Phi}^{(i)} dx = \bar{0}$$

$$\text{Or } \sum_{i=1}^n \left([A^{(i)}] \bar{y}^{(i)} - \bar{F}^{(i)} \right) = \bar{0}$$

$$\text{where } [A^{(i)}] = \int_{x_{2i-1}}^{x_{2i+1}} (p \bar{\Phi}^{(i)'} \bar{\Phi}^{(i)'}{}^T + q \bar{\Phi}^{(i)} \bar{\Phi}^{(i)T}) dx, \quad \bar{F}^{(i)} = \int_{x_{2i-1}}^{x_{2i+1}} f(x) \bar{\Phi}^{(i)} dx$$

Assembling the equations of all elements yields

$$[A]\{Y\} = \{F\}, \quad \text{where } \bar{Y} = \{y_0 \ \dots \ y_{2n+1}\}^T$$

Omitting the first row, first column, the last row and the last column of $[A]$, the first entry and the last entry of $\{F\}$, and the first entry and the last entry of $\{Y\}$, simultaneously, yields

$$[\tilde{A}]\{\tilde{Y}\} = \{\tilde{F}\}, \text{ where } \{\tilde{Y}\} = \{y_1 \dots y_{2n}\}^T.$$

The values at the interior node can be obtained from the above equation.

HW 11

Given the equation $y'' = -(x+1)y' + 2y + (1-x^2)e^{-x}$, $0 \leq x \leq 1$, $y(0) = 1$,

$$y(1) = 2$$

use $h = 0.1$

Questions:

- Use the shooting method to approximate the solution of the problem
- Use the finite-difference method to approximate the solution
- Use the variation approach to approximate the solution.